Equality of Domination Number and Isolate Domination Number in Some Graphs

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Abstract

In this paper, the equality and strong equality properties among domination number, total domination number and isolate domination number are studied. Especially, it is proved that generalized Petersen graph GP(n, k), Cocktail Party graph $K_{n\times 2}$, Crown graph S_n^0 and Barbell graph G have the property $\gamma_0 = \gamma$.

Keywords: Domination, Isolate domination, Total domination, Strong equality, Generalized Petersen graph, Cocktail Party graph, Crown graph, Barbell graph

Introduction

One of the fastest growing areas within graph theory is the study of domination. In 1978, Cockayne first defined that has now become a well-known inequality chain of domination related parameters of a graph G as follows:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G), \quad (1)$$

where ir(G) and IR(G) denote the lower and upper irredundance numbers, $\gamma(G)$ and $\Gamma(G)$ denote the lower and upper domination numbers and i(G) and $\beta_0(G)$ denote the independent domination number and independence number of a graph G, respectively. This equality chain has become one of the strongest focal points for doing research in domination theory. More specifically, extending this chain in either side by fundamental domination parameters is one such direction of the research. By fundamental domination parameters, it means that they can be defined for all nontrivial connected graphs with neither loops nor multiple edges. Following this, such domination parameters namely isolate domination number $\gamma_0(G)$ and upper isolated domination number $\Gamma_0(G)$ were introduced in [Hayne, 1998], where the existing domination chain (1) was extended as follows:

$$ir(G) \leq \gamma(G) \leq \gamma_0(G) \leq i(G) \leq \beta_0(G) \leq \Gamma_0(G) \leq \Gamma(G) \leq IR(G). \quad (2)$$

This type of domination is defined as follows: A dominating set S of a graph G is said to be an **isolate dominating set** of G if the sub graph induced by S has at least one isolated vertex. An isolate dominating set S is said to be minimal isolate dominating set if no proper subset of S is an isolate dominating set. The minimum and maximum cardinality of a minimal isolate dominating set of G are called the **isolate domination number** $\gamma_0(G)$ and the upper isolate domination number $\Gamma_0(G)$ respectively.

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Necessary and Sufficient Conditions for a Graph with $\gamma_0(G) = \gamma(G)$

In this section, some properties of a graph G with $\gamma_0(G) = \gamma(G)$ are derived.

Proposition 1

For a graph G, $\gamma_0(G) = \gamma(G)$ if and only if there is a $\gamma(G)$-set S such that G[S] has some isolated vertex.

Proof:

Let G be a graph and $\gamma_0(G) = \gamma(G)$. Clearly, there is a $\gamma(G)$-set S such that G[S] has some isolated vertex. Conversely, suppose that there is a $\gamma(G)$-set S such that G[S] has some isolated vertex. Hence S is also a minimum isolate dominating set. Therefore, $\gamma_0(G) = \gamma(G)$. $\square$

The following follows from Proposition 1.

Corollary 2

For a graph G, $\gamma_0(G) \neq \gamma(G)$ if and only if every minimum dominating set is a total dominating set.

Proof:

Let G be a graph. Suppose that $\gamma_0(G) \neq \gamma(G)$. By Proposition 1, there is no γ(G)-set S such that G[S] has some isolated vertex. Therefore, every minimum dominating set is a total dominating set. Conversely, suppose that every minimum dominating set ($\gamma(G)$-set) is a total dominating set. Hence, there is no $\gamma(G)$-set such that the sub graph induced by it has no isolated vertex. Therefore, by Proposition 2.1, $\gamma_0(G) \neq \gamma(G)$. $\square$

From Corollary 2, we obtain the following.

Theorem 3

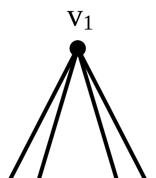
For a graph G, $\gamma_0(G) = \gamma(G)$ if and only if $\gamma(G) \not\equiv \gamma_t(G)$.

Proof:

Let G be a graph and $\gamma(G) = \gamma_0(G)$. By Corollary 2, every minimum dominating set is not a total dominating set. Therefore, $\gamma(G) \not\equiv \gamma_t(G)$. Conversely, suppose that $\gamma(G) \not\equiv \gamma_t(G)$. Then, every minimum dominating set is not a total dominating set. By Corollary 2, $\gamma(G) = \gamma_0(G)$. $\square$

Example

In Figure 1, it is presented the graph G and it cannot determine every minimum dominating set of G and it is a total dominating set of G. Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.



G:

Figure 1 A graph G

First we determine the dominating sets S_1, S_2, S_3 and S_4, where $S_1 = \{v_1, v_2\}$, $S_2 = \{v_4, v_5\}$, $S_3 = \{v_2, v_4\}$ and $S_4 = \{v_1, v_6\}$. They all are minimum dominating sets of G. So, $\gamma(G) = 2$. The sets S_1 and S_2 are $\gamma_t(G)$-sets. The sets S_3 and S_4 are $\gamma_0(G)$-sets but they are not $\gamma_t(G)$-set. Clearly, every minimum dominating set is not a total dominating set. Therefore, $\gamma(G) \not\equiv \gamma_t(G)$. Hence, it satisfies the above theorem.

The following result provides a necessary condition for $\gamma(G) \equiv \gamma_t(G)$ in a graph G.

Lemma 5

For any graph G, if $\gamma(G) \equiv \gamma_t(G)$, then every vertex in any $\gamma(G)$-set S has at least two external private neighbors.

Proof:

Let G be a graph with $\gamma(G) \equiv \gamma_t(G)$, and consider a $\gamma(G)$-set S. Since $\gamma(G) \equiv \gamma_t(G)$, S is also a $\gamma_t(G)$-set implying that G[S] has no isolated vertices. Hence, each vertex in S has an external private neighbor. Suppose $v \in S$ and epn(v, S) $\cap$ (V − S) = {u} where $u \in V - S$. Then, $\{u\} \cup (S - \{v\})$ is a $\gamma(G)$-set that is not a $\gamma_t(G)$-set, contradicting the fact that $\gamma(G) \equiv \gamma_t(G)$. Hence, every vertex in S has at least two external private neighbors. □

An immediate consequence of Lemma 5 is as follows.

Corollary 6

If G is a graph of order n satisfying $\gamma(G) \equiv \gamma_t(G)$, then $\gamma(G) \le \dfrac{n}{3}$.

Proof:

Suppose that $\gamma(G) \equiv \gamma_t(G)$. By Lemma 5, every vertex in any $\gamma(G)$-set S has at least two external private neighbors. Thus $|V - S| \geq \frac{2n}{3}$. Therefore, $|S| \leq \frac{n}{3}$. □

Proposition 7

If $\gamma_0(G) \neq \gamma(G)$, then $\gamma(G) = \gamma_t(G) \leq \frac{n}{3}$ and $|epn(x, S)| \geq 2$ for any $\gamma(G)$-set S and any vertex $x \in S$.

Proof:

Suppose that $\gamma_0(G) \neq \gamma(G)$. By Theorem 3, $\gamma(G) \equiv \gamma_t(G)$. By Lemma 5 and Corollary 6, $\gamma(G) = \gamma_t(G) \leq \frac{n}{3}$ and $|epn(x, S)| \geq 2$ for any $\gamma(G)$-set S, and any vertex $x \in S$. □

Proposition 8

For any graph G, if diam (G) = 2 and $\gamma(G) \neq \gamma_t(G)$, then $\gamma_0(G) = \gamma(G)$.

Proof:

Let G be a graph with diam (G) = 2. Let S be $\gamma(G)$-set and u, $v \in S$. We consider two cases.

Case (i)

Let $d_{G[S]}(u, v) > 2$. Since $d_G(u, v) \leq 2$, there exists a vertex $w \in V - S$ such that w is adjacent to both u and v. Therefore u and v have no external private neighbor. Thus, by Proposition 2.7, $\gamma_0(G) = \gamma(G)$.

Case (ii)

Let $d_{G[S]}(u, v) \leq 2$. Every minimum dominating set is not a total dominating set. Therefore, by Proposition 2.7, $\gamma_0(G) = \gamma(G)$. □

Theorem 9

If either $\delta(G) = 0$ or $\Delta(G) = n - 1$, then $\gamma_0(G) = \gamma(G)$, where n is the order of G.

Proof:

Obviously, when $\delta(G) = 0$, the isolated vertices of G will be in every γ-set of G, and hence $\gamma_0(G) = \gamma(G)$. Also when $\Delta(G) = n - 1$, $\gamma(G) = 1$, and hence the result follows. □

Some Special Graphs with $\gamma_0(G) = \gamma(G)$

Theorem 10

For any claw-free graph G, $\gamma_0(G) = \gamma(G)$.

Proof:

Let G be a claw-free graph. Suppose that $\gamma_0(G) \neq \gamma(G)$. Let S be a $\gamma(G)$-set. By Proposition 2.7, $\gamma(G) = \gamma_t(G)$, and |epn (u, S)| ≥ 2 for any vertex $u \in S$. Let x be a vertex of S with $\deg_{G[S]}(x) = 1$, and $y \in N(x) \cap S$. If G[epn(x,S)] is a complete graph, then $(S - \{x\}) \cup \{w\}$ is a $\gamma(G)$-set which is not a total dominating set, where $w \in$ epn(x, S). This contradicts Corollary 2. Thus G[epn(x, S)] is not a complete graph. Since y is adjacent to no vertex of epn (x, S), and |epn (x, S)| ≥ 2, it is found that G contains a claw, a contradiction. Therefore, $\gamma_0(G) = \gamma(G)$. □

Example

In Figure 2, we consider the claw-free graph G and determine the domination number and isolate domination number. Let V(G) = $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$. We consider $D_1 \subset V$ such that $D_1 = \{v_2, v_6, v_{10}\}$ and $D_2 \subset V$ such that $D_2 = \{v_4, v_7, v_{11}\}$. Both are minimum dominating sets and $G[D_1]$ and $G[D_2]$ have isolated vertices. Therefore, $\gamma(G) = \gamma_0(G) = 3$.

G:

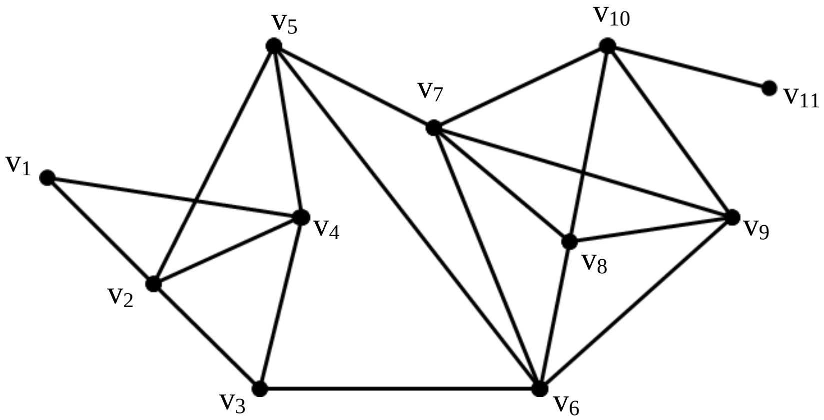


Figure 2 A claw-free graph G

Theorem 11

For a generalized Petersen graph G, $\gamma_0(G) = \gamma(G)$.

Proof:

Let S be a γ-set of G. If G[S] has an isolated vertex then we are done. Now, assume that G[S] has no isolated vertices. Then, every vertex of S will have at least one external private neighbor. Further, it can be observed that the cardinality of S will be minimum when each vertex of S has exactly two external private neighbors and also in that case, $|S| = \frac{n}{3}$. Now, we assert that there is an isolate dominating set for G with cardinality less than $\frac{n}{3}$. By bearing in mind the usual structure of the generalized Petersen graph, we label the vertices of the outer circle as $u_1, u_2, ..., u_{\frac{n}{2}}$ and the inner circle as $v_1, v_2, ..., v_{\frac{n}{2}}$ such that $(u_i, v_i) \in E(G)$, for every i. Now, the set $S' = \{u_1, v_3, u_5, v_7, ...\}$, where the last vertex of S' belongs to the set $\{u_{\frac{n}{2}}, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, v_{\frac{n}{2}-1}\}$, forms an isolate dominating set of G. Therefore $|S'| = \left\lceil \frac{n}{4} \right\rceil \le \frac{n}{3}$. Thus, $\gamma(G) \ge \gamma_0(G)$. But $\gamma(G) \le \gamma_0(G)$. Therefore, $\gamma_0(G) = \gamma(G)$. □

Example

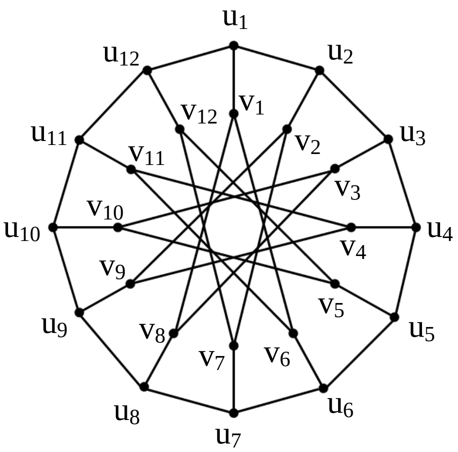


Figure 3 Generalized Petersen graph GP(12, 5)

In this example, we consider the generalized Petersen graph GP(12, 5) in Figure 3. For the result of the above theorem, first determining its domination number and isolate domination number are done. Let $V(G) = \{u_1, ..., u_{12}, v_1, ..., v_{12}\}$. The set $S = \{u_1, v_3, u_5, v_7, u_9, v_{11}\}$ is a minimum dominating set and also a minimum isolate dominating set. Therefore, $\gamma(G) = \gamma_0(G) = 6$.

Theorem 12

For any Cocktail Party graph $K_{n\times 2}$, $\gamma(K_{n\times 2}) = \gamma_0(K_{n\times 2})$.

Proof:

The diameter of any Cocktail Party graph $K_{n\times 2}$ is 2. Therefore by Proposition 2.8, the domination number and isolate domination number of $K_{n\times 2}$ is equal. That is, $\gamma(K_{n\times 2}) = \gamma_0(K_{n\times 2})$. Let V_1 and V_2 be the vertex sets of $K_{n\times 2}$ respectively. Then any pair of vertices $\{u_i, v_i\}$, for any integer i, where $u_i \in V_1$ and $v_i \in V_2$ is a minimum dominating set as well as a minimum isolate dominating set with cardinality 2. Hence $\gamma(K_{n\times 2}) = \gamma_0(K_{n\times 2}) = 2$. □

Theorem 13

For any Crown graph S_n^0, $\gamma(S_n^0) = \gamma_0(S_n^0)$.

Proof:

A Crown graph S_n^0 is equivalent to the complete bipartite graph $K_{n,n}$ when the horizontal edges of $K_{n,n}$ are removed. Let X and Y be two partite sets of vertices of S_n^0. Then any pair of vertices $\{x_i, y_i\}$, for any integer i, where $x_i \in X$ and $y_i \in Y$ is a minimum dominating set as well as an isolate dominating set with cardinality 2. Hence, $\gamma(S_n^0) = \gamma_0(S_n^0) = 2.$

Theorem 14

For any n-Barbell graph G, $\gamma(G) = \gamma_0(G)$.

Proof:

Let G be an n-Barbell graph and $V_1 = \{u_1, ..., u_n\}$ and $V_2 = \{v_1, ..., v_n\}$ be two vertex sets of an n-barbell graph G. Then any pair of vertices $\{u_i, v_i\}$, for any integer i, where $u_i \in V_1$ and $v_i \in V_2$ is a minimum dominating set with cardinality 2. But every minimum dominating set is an isolate dominating set except the minimum dominating set formed by end vertices of the bridge. Hence, by Theorem 2.3, $\gamma(G) = \gamma_0(G) = 2.$

Conclusion

The domination number and isolate domination number of Generalized Petersen graph, Claw-free graph, Cocktail Party graph, Crown graph and Barbell graph are equal. It is found that the isolate domination numbers of any Cocktail Party graph, Crown graph and Barbell graph are exactly 2.

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